A class of well-posed approximations for constrained second order hyperbolic equations

Yves Renard

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Abstract

The purpose of this paper is to present a new family of numerical methods for the approximation of second order hyperbolic partial differential equations submitted to a convex constraint on the solution. The principle is a singular modification of the mass matrix obtained by the mean of different discretizations of the solution and of its time derivative. The major interest of these methods is that the semi-discretized problem is well-posed and energy conserving. Numerical experiments show that this is a crucial property to build stable numerical schemes.

Keywords: hyperbolic partial differential equation, constrained equation, finite element methods, variational inequalities.

Introduction

An interesting class of hyperbolic partial differential equations with constraints on the solution consists in elastodynamic contact problems for which the vast majority of traditional numerical schemes show spurious oscillations on the contact displacement and stress (see for instance [8, 5, 6]). Moreover, these oscillations do not disappear when the time step decreases. Typically, they have instead tended to increase. This is a characteristic of order two hyperbolic equations with unilateral constraints that makes it very difficult to build stable numerical schemes. These difficulties have already led to many research under which a variety of solutions were proposed. Some of them consists in adding damping terms (see [19] for instance), but with a loss of accuracy on the solution, or to implicit the contact stress [2] but with a loss of kinetic energy which could be independent of the discretization parameters (see the numerical experiments). Some energy conserving schemes have also been proposed in [7, 20, 12, 11, 5, 6]. Unfortunately, these schemes, although more satisfactory than the most other schemes, lead to large oscillations on the contact stress. Besides, most of them do not strictly respect the constraint.

In this paper, we propose a new class of methods whose principle is to make different approximations of the solution and of its time derivative. Compared to the classical space semi-discretization, this corresponds to a singular modification of the mass matrix. In this sense, it is in the same class of methods than the mass redistribution method proposed in [8, 9] which is more specifically adapted to problems with constraints on the boundary. The main feature is to provide a well-posed space semi-discretization. The numerical tests show that it has a crucial influence on the stability of standard scheme and on the quality of the approximation, especially for the computation of Lagrange multipliers corresponding to the constraints.

Indeed, the classical semi-discretizations, for example with finite element methods, give a problem in time which is a measure differential inclusion (see [14, 15, 16, 17]). Such a

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differential inclusion is systematically ill-posed, unless an additional impact law is considered for each contact node. However, the scheme obtained with the addition of an impact law in [16] leads also to spurious oscillations.

The semi-discretization we propose here leads to a problem which is equivalent to a regular Lipschitz ordinary differential equation. Thus, time integration schemes at least converge for a fixed space discretization when the time step tends to zero. This work generalizes in a sense the methods presented in [9, 4].

The outline of the paper is the following. Section 1 is devoted to the description of the abstract hyperbolic equation with constraints and the equivalent variational inequality. Section 2 presents the new approximation methods and the main results of well-posedness and energy conservation. Section 3 briefly introduces the fully discrete problem obtained with the finite difference midpoint scheme. Then, in section 4, a non-trivial model problem which corresponds for instance to the dynamics of a thin membrane under an obstacle condition is developed. An example of well-posed discretization is also built in this section. Finally, section 5 presents some numerical experiments on this model which shows in particular that the midpoint scheme is stable with well-posed semi-discretizations and unstable otherwise.

1 The abstract hyperbolic equation

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $H = L^2(\Omega)$ the standard Hilbert space of square integrable functions on Ω. Let W be a Hilbert space such that

$$W \subset H \subset W',$$

with dense compact and continuous inclusions and let

$$A : W \to W'$$

be a linear elliptic continuous operator, i.e. which satisfies

$$\exists \alpha > 0, \quad \forall w \in W, \quad \langle Aw, w \rangle_{W',W} \geq \alpha \|w\|_W^2, \qquad \exists c > 0, \quad \forall w \in W, \quad \|Aw\|_{W'} \leq c\|w\|_W.$$

We consider the following problem

$$\left\{ \begin{array}{l} \text{Find } u : [0,T] \to K \text{ such that} \\ \dfrac{\partial^2 u}{\partial t^2}(t) + Au(t) \in f - N_K(u(t)) \quad \text{for a.e. } t \in \ (0,T], \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = v_0, \end{array} \right. \tag{1}$$

where K is a closed convex nonempty subset of W, $f \in W'$, $u_0 \in K$, $v_0 \in H$, $T > 0$ and $N_K(u)$ is the normal cone to K defined by (see [1] for instance)

$$N_K(u) = \left\{ \begin{array}{l} \emptyset \text{ if } u \notin K, \\ \{f \in W' : \langle f, w-u \rangle_{W',W} \leq 0, \quad \forall w \in K\} \text{ if } u \in K. \end{array} \right.$$

This means that $u(t)$ satisfies the second order hyperbolic equation and is constrained to remain in the convex K. As far as we know, there is no general result of existence and uniqueness for the solution to this kind of equation. Some existence results for a scalar Signorini problem can be found in [13, 10]. Introducing now the linear and bilinear maps

$$l(v) = \langle f, v \rangle_{W',W}, \quad a(u,v) = \langle Au, v \rangle_{W',W}$$

Problem (1) can be rewritten as the following variational inequality:

$$\left\{\begin{array}{l} \text{Find } u:[0,T]\to K \text{ such that for a.e. } t\in\ (0,T],\\ \langle\dfrac{\partial^2 u}{\partial t^2}(t), w-u(t)\rangle_{W',W} + a(u(t),w-u(t)) \geq l(w-u(t)) \ \ \forall w\in K,\\ u(0)=u_0,\quad \dfrac{\partial u}{\partial t}(0)=v_0. \end{array}\right. \tag{2}$$

Note that the terminology "variational inequality" is used here in the sense that Problem (1) derives from the conservation of the energy functional

$$J(t)=\frac{1}{2}\int_\Omega (\frac{\partial u}{\partial t}(t))^2dx+\frac{1}{2}a(u(t),u(t))-l(u(t))+I_K(u(t)),$$

where $I_K(u(t))$ is the convex indicator function of K. However, it is generally not possible to prove that each solution to Problem (2) is energy conserving, due to the weak regularity involved.

2 Approximation and well-posedness result

The goal of this section is to present well-posed space semi-discretizations of Problem (2). The strategy adopted is to use a Galerkin method with different approximations of u and of $v=\dfrac{\partial u}{\partial t}$. Let W^h and H^h be two finite dimensional vector subspaces of W and H respectively. Let $K^h\subset W$ be a closed convex nonempty approximation of K. The proposed approximation of Problem (2) is the following

$$\left\{\begin{array}{l} \text{Find } u^h:[0,T]\to K^h \text{ and } v^h:[0,T]\to H^h \text{ such that}\\ \displaystyle\int_\Omega \frac{\partial v^h}{\partial t}(w^h-u^h)dx + a(u^h,w^h-u^h)\geq l(w^h-u^h)\ \ \forall w^h\in K^h,\ \ \forall t\in\ (0,T],\\ \displaystyle\int_\Omega (v^h-\frac{\partial u^h}{\partial t})q^h dx=0\ \ \forall q^h\in H^h,\ \ \forall t\in\ (0,T],\\ u^h(0)=u_0^h,\quad v^h(0)=v_0^h, \end{array}\right. \tag{3}$$

where $u_0^h\in K^h$ and $v_0^h\in H^h$ are some approximations of u_0 and v_0 respectively. Of course, when $H^h=W^h$ this corresponds to a standard Galerkin approximation of Problem (2).

Let φ_i, $1\leq i\leq N_W$ and ψ_i, $1\leq i\leq N_H$ be some basis of W^h and H^h respectively, and let the matrices A, B and C, of sizes $N_W\times N_W$, $N_H\times N_W$ and $N_H\times N_H$ respectively, and the vectors L, U and W, of size N_W, N_W and N_H respectively, be defined by

$$A_{i,j}=a(\varphi_i,\varphi_j),\quad B_{i,j}=\int_\Omega \psi_i\varphi_j dx,\quad C_{i,j}=\int_\Omega \psi_i\psi_j dx,$$

$$L_i=l(\varphi_i),\quad u^h=\sum_{i=1}^{N_W}U_i\varphi_i,\quad v^h=\sum_{i=1}^{N_H}V_i\psi_i.$$

Then, the expression of Problem (3) in terms of vectors and matrices is the following:

$$\left\{\begin{array}{l} \text{Find } U:[0,T]\to \overline{K}^h \text{ and } V:[0,T]\to\mathbb{R}^{N_H} \text{ such that } \forall t\in\ (0,T],\\ (W-U(t))^T(B^T\dot{V}(t)+AU(t))\geq (W-U(t))^T L,\quad \forall W\in\overline{K}^h,\\ CV(t)=B\dot{U}(t),\\ U(0)=U_0,\quad V(0)=V_0. \end{array}\right. \tag{4}$$

where $\dot{U}(t)$ and $\dot{V}(t)$ denote the derivative with respect to t of $U(t)$ and $V(t)$ respectively and $\overline{K}^h$ is defined by

$$\overline{K}^h = \{W \in \mathbb{R}^{N_W} : \sum_{i=1}^{N_W} W_i\varphi_i \in K^h\}.$$

Now, the unknown V can be eliminated since C is always invertible which leads to the relation $V(t) = C^{-1}B\dot{U}(t)$. Thus denoting

$$M = B^T C^{-1} B,$$

Problem (4) can be rewritten

$$\begin{cases} \text{Find } U : [0,T] \to \overline{K}^h \text{ such that} \\ (W - U(t))^T (M\ddot{U}(t) + AU(t)) \geq (W - U(t))^T L, \quad \forall W \in \overline{K}^h,\ \forall t \in\ (0,T], \\ U(0) = U_0, \quad B\dot{U}(0) = CV_0. \end{cases} \tag{5}$$

Remark 1 *If the couple of discretization spaces H^h, W^h satisfies a classical inf-sup condition then the matrix B is surjective and the initial condition $B\dot{U}(0) = CV_0$ is always admissible. Conversely, if B is not surjective then the initial condition V_0 has to satisfy the following condition*

$$V_0 \in \mathrm{Im}(C^{-1}B). \tag{6}$$

In fact, this condition is also implicitly contained in Problem (4).

In comparison with the standard approximation where $H^h = W^h$ the presented method only replace the standard mass matrix $\left(\int_\Omega \varphi_i\varphi_j dx\right)_{i,j}$ by $M = B^T C^{-1} B$. In the interesting cases where $\dim(H^h) < \dim(W^h)$ this corresponds to replace the standard invertible mass matrix by a singular one. We propose to call this kind of method a singularly modified mass matrix method (S4M). Of course, the numerical implementation will be facilitated when the matrix C is diagonal. This is the case for instance when H^h is defined with P_0 finite element method or with a more general finite element method using an adapted sub-integration (lumped mass matrix). We will see how, rather surprisingly, the introduction of a singular mass matrix allows to recover the well-posedness of the approximation.

The goal now is to give a sufficient condition for Problem (5) (or equivalently Problem (3) or (4)) to be well-posed. To this end, we will define a more restrictive framework (see the concluding remarks for a possibility to extend this framework). We will suppose that K^h is defined by

$$K^h = \{w^h \in W^h : g^i(w^h) \leq \alpha^i, 1 \leq i \leq N_g\},$$

where $\alpha^i \in \mathbb{R}$ and $g^i : W^h \to \mathbb{R}$, $1 \leq i \leq N_g$ are some linear independent maps. Of course, this restricts the possibilities concerning the convex K since K^h is supposed to be an approximation of K. With vector notations this leads to

$$\overline{K}^h = \{W \in \mathbb{R}^{N_W} : (G^i)^T W \leq \alpha_i, 1 \leq i \leq N_g\},$$

where $G^i \in \mathbb{R}^{N_W}$ are such that $g^i(w^h) = (G^i)^T W$, $1 \leq i \leq N_g$. We will also denote by G the $N_W \times N_g$ matrix whose components are

$$G_{ij} = (G^i)_j.$$

Let us consider the subspace F^h of W^h defined by

$$F^h = \{w^h \in W^h : \int_\Omega w^h q^h = 0 \ \ \forall q^h \in H^h\}.$$

Then, the corresponding set $F = \{W \in \mathbb{R}^{N_W} : \sum_{i=1}^{N_W} W_i \varphi_i \in F^h\}$, is such that

$$F = \mathrm{Ker}(B).$$

In this framework, we will prove that the following condition is sufficient for the well-posedness of the discrete problem (5):

$$\inf_{\substack{Q \in \mathbb{R}^{N_g} \\ Q \neq 0}} \sup_{\substack{W \in F \\ W \neq 0}} \frac{Q^T G W}{|Q| \ |W|} > 0, \tag{7}$$

where $|Q|$ and$|W|$ stands for the Euclidean norm of Q in $\mathbb{R}^{N_g}$ and W in $\mathbb{R}^{N_W}$ respectively. This condition is equivalent to the fact that the linear maps g^i are independent on F^h and also to the fact that G is surjective on F. A direct consequence is that it implies $\dim(F^h) \geq N_g$, and consequently

$$\dim(H^h) \leq \dim(W^h) - N_g.$$

This again prescribed some conditions on the approximation made which links W^h, H^h and also K^h. We will see on Section 4 that this condition can be satisfied for interesting practical situations. We can now prove the following result.

Theorem 1 *If W^h, H^h and K^h satisfy the condition (7) then Problem (5) admits a unique solution. Moreover, this solution is Lipschitz-continuous with respect to t.*

First, let us establish the following intermediary result:

Lemma 1 *If W^h, H^h and K^h satisfy the condition (7) then there exists F^c a sub-space of $\mathbb{R}^{N_W}$ such that $F^c \subset Ker(G)$ and such that F and F^c are complementary sub-spaces.*

Proof. For $W \in \mathbb{R}^{N_W}$ let $X_F \in F$ be such that

$$G(X_F) = G(W).$$

Such an X_F exists since a consequence of condition (7) is that the matrix G defines a surjective linear map from F to $\mathbb{R}^{N_g}$. Thus

$$W = (W - X_F) + X_F,$$

is a decomposition of W with $W - X_F \in \mathrm{Ker}(G)$ and $X_F \in F$. This proves that $\mathbb{R}^{N_W} = F + \mathrm{Ker}(G)$. The result of the lemma is then a consequence of the basis extension theorem. □

Proof of Theorem 1. Now, using the result of Lemma 1, let us decompose $U, W \in \mathbb{R}^{N_W}$ as

$$U = U_F + U_{F^c}, \quad W = W_F + W_{F^c},$$

with $U_F, W_F \in F$ and $U_{F^c}, W_{F^c} \in F^c$. The inequation of (5) can be written for all $t \in \ (0, T]$

$$\begin{aligned}(W_{F^c} - U_{F^c})^T (M\ddot{U}_{F^c} + AU_{F^c} + AU_F) + (W_F - U_F)^T (AU_{F^c} + AU_F) \\ \geq (W_{F^c} - U_{F^c})^T L + (W_F - U_F)^T L, \quad \forall W_F \in \overline{K}^h \cap F, \ \ \forall W_{F^c} \in F^c.\end{aligned} \tag{8}$$

Taking now $W_{F^c} = U_{F^c}$ one obtains

$$(W_F - U_F)^T AU_F \geq (W_F - U_F)^T (L - AU_{F^c}), \quad \forall W_F \in \overline{K}^h \cap F. \tag{9}$$

This is a variational inequality for the unknown U_F. The solution to this variational inequality minimizes the quadratic functional

$$J_F(W_F) = \frac{1}{2} W_F^T AW_F - W_F^T (L - AU_{F^c})$$

over the closed convex $\overline{K}^h \cap F$. The ellipticity assumption implies that the matrix A is coercive. This leads to the existence and uniqueness of the solution U_F to this variational inequality due to the Stampacchia theorem. Moreover, this is a classical result that U_F depends Lipschitz-continuously on U_{F^c}. Indeed, let U_F^1 and U_F^2 be two solutions for $U_{F^c}^1$ and $U_{F^c}^2$ respectively. Then it can be straightforwardly deduced from the variational inequality that

$$(U_F^2 - U_F^1)^T A(U_F^2 - U_F^1) \leq (U_F^2 - U_F^1)^T A(U_{F^c}^2 - U_{F^c}^1).$$

Thus due to the coercivity of the matrix A one obtain for $c > 0$ a generic constant

$$|U_F^2 - U_F^1| \leq c|U_{F^c}^2 - U_{F^c}^1|.$$

We will thus use the notation $U_F(U_{F^c})$. Now, since inequation (8) has to be satisfied for all $W_{F^c} \in F^c$, this implies that $U_{F^c}(t)$ verifies for all $t \in \ (0, T]$

$$W_{F^c}^T M\ddot{U}_{F^c} = W_{F^c}^T (L - AU_{F^c} - AU_F(U_{F^c})) \quad \forall W_{F^c} \in F^c, \tag{10}$$

which represents an ordinary differential equation with Lipschitz-continuous right-hand side. Since the matrix M is nonsingular on F^c (because F^c is complementary to $F = \mathrm{Ker}(B)$ and $M = B^T C^{-1} B$) there exists a unique solution to the associated initial value problem with the initial conditions $U_{F^c}(0) = (U_0)_{F^c}$ and $B\dot{U}_{F^c}(0) = CV_0$ assuming condition (6) when B is not surjective.

Since $U_{F^c}(t)$ is the solution to a second order autonomous ordinary differential equation with Lipschitz-continuous right-hand side, it has at least the regularity $U_{F^c} \in W^{3,\infty}(0, T; F^c)$. Finally, the whole $U(t)$ is Lipschitz-continuous with respect to t due to the fact that U_F depends Lipschitz-continuously on U_{F^c}. □

Now, an interesting property is that the solution to Problem (5) satisfies the so-called persistency condition (see [11, 12]). This is a condition between $\dot{U}(t)$ and the Lagrange multipliers corresponding to the constraints. In a sense, this is a stronger condition than the so-called complementary condition which links $U(t)$ and the Lagrange multipliers. In fact, Problem (5) can be re-written

$$\begin{cases} \text{Find } U : [0, T] \to \overline{K}^h \text{ such that} \\ M\ddot{U}(t) + AU(t) \in L - N_{\overline{K}^h}(U(t)) \quad \forall t \in \ (0, T], \\ U(0) = U_0, \quad B\dot{U}(0) = CV_0, \end{cases} \tag{11}$$

where $N_{\overline{K}^h}(U(t))$ is the normal cone to $\overline{K}^h$. A straightforward computation leads to the following result:

$$N_{\overline{K}^h}(U(t)) = \begin{cases} \emptyset \text{ if } U(t) \notin \overline{K}^h, \\ \left\{ \displaystyle\sum_{\substack{1 \leq i \leq N_g \\ (G^i)^T U(t) < \alpha_i}} \mu^i G^i : \mu_i \geq 0 \right\} \quad \text{if } U(t) \in \overline{K}^h. \end{cases}$$

Thus, introducing Lagrange multipliers, the discrete problem is also equivalent to the following one:

$$\begin{cases} \text{Find } U:[0,T]\to \overline{K}^h \text{ and } \lambda^i:[0,T]\to\mathbb{R}, 1\le i\le N_g \text{ such that } \forall t\in\ (0,T] \\ M\ddot{U}(t)+AU(t)=L+\sum_{i=1}^{N_g}\lambda^i(t)G^i, \\ \lambda^i(t)\le 0, \quad (G^i)^T U(t)-\alpha_i\le 0, \quad \lambda^i(t)((G^i)^T U(t)-\alpha_i)=0, \quad 1\le i\le N_g, \\ U(0)=U_0, \quad B\dot{U}(0)=CV_0, \end{cases} \tag{12}$$

Proposition 1 *If W^h, H^h and K^h satisfy the condition (7) then the solution $U(t)$ to Problem (5) verifies the following persistency condition*

$$\lambda^i(t)(G^i)^T\dot{U}(t)=0 \quad \forall t\in\ (0,T], \quad 1\le i\le N_g.$$

Proof. With still the same decomposition as the one in Theorem 1 we deduce from (12) that $\lambda^i,\ 1\le i\le N_g$ satisfy

$$W_F^T AU_F = W_F^T(L-AU_{F^c}+\sum_{i=1}^{N_g}\lambda^i G^i), \quad \forall W_F\in F.$$

Since U_F depends Lipschitz-continuously on U_{F^c}, this equation implies that each λ^i depends also Lipschitz-continuously on U_{F^c}. Thus each $\lambda^i(t)$ is Lipschitz-continuous with respect to t. But

$$\lambda^i=0 \text{ on } \mathrm{Supp}((G^i)^T U-\alpha_i)=\omega^i\subset[0,T], \quad 1\le i\le N_g,$$

where Supp(f) denotes the support of the function $f(t)$. The continuity of $\lambda^i(t)$ implies

$$\lambda^i=0 \text{ on } \overline{\omega^i}.$$

Since $(G^i)^T U-\alpha_i=0$ on the complementary of ω_i, then its derivative $(G^i)^T\dot{U}$ vanishes also on the interior of complementary of ω_i which proves the result of the proposition. □

Now, we can prove that the persistency condition implies the energy conservation.

Theorem 2 *If W^h, H^h and K^h satisfy the condition (7) then the solution $U(t)$ to Problem (5) is energy conserving in the sense that the discrete energy*

$$J^h(t)=\frac{1}{2}\dot{U}^T(t)M\dot{U}(t)+\frac{1}{2}U^T(t)AU(t)-U^T(t)L,$$

is constant with respect to t.

Proof. The first equation of (12) implies

$$\dot{U}^T M\ddot{U}(t)+\dot{U}^T AU(t)=\dot{U}^T L+\sum_{i=1}^{N_g}\dot{U}^T\lambda^i(t)G^i, \quad \text{on } [0,T].$$

Integrating from 0 to t and using Proposition 1 one can conclude that

$$\frac{1}{2}\dot{U}^T(t)M\dot{U}(t)+\frac{1}{2}U^T(t)AU(t)-U^T(t)L=J^h(0).$$

□

3 Full discretization with a midpoint scheme

The midpoint scheme is interesting since it is energy conserving on the linear part (equation without constraint). A midpoint scheme on Problem (5) has the following expression:

$$\left\{\begin{array}{l} U^0 \text{ and } V^0 \text{ be given. For } n \geq 0 \\ (W - U^{n+\frac{1}{2}})^T (MZ^{n+\frac{1}{2}} + AU^{n+\frac{1}{2}}) \geq (W - U^{n+\frac{1}{2}})^T L, \quad \forall W \in \overline{K}^h, \\ U^{n+\frac{1}{2}} = \dfrac{U^n + U^{n+1}}{2}, \quad V^{n+\frac{1}{2}} = \dfrac{V^n + V^{n+1}}{2}, \\ U^{n+1} = U^n + \Delta t V^{n+\frac{1}{2}}, \\ V^{n+1} = V^n + \Delta t Z^{n+\frac{1}{2}}. \end{array}\right. \tag{13}$$

With U^n and V^n be given, U^{n+1} is then solution to

$$(W - \frac{U^n + U^{n+1}}{2})^T (\frac{2}{\Delta t^2} MU^{n+1} + \frac{1}{2} AU^{n+1}) \geq (W - \frac{U^n + U^{n+1}}{2})\tilde{L}, \quad \forall W \in \overline{K}^h,$$

where $\tilde{L}$ depends on U^n and V^n. Due to the coercivity of the matrix A, this variational inequality always admits a unique solution, whatever is the choice of W^h, H^h and K^h (even with a standard discretization, i.e. in the case $W^h = H^h$). Note that the well-posedness of (13) does nothing to the overall stability of the scheme.

4 A model problem

The goal of this section is to provide a simple but interesting situation for which some consistent approximations satisfy the condition (7). With $W = H^1(\Omega)$ and $K = \{w \in W : w \geq 0 \text{ a.e. on } \Omega\}$ we consider the following problem:

$$\left\{\begin{array}{l} \text{Find } u : [0,T] \to K \text{ such that} \\ \dfrac{\partial^2 u}{\partial t^2}(t) - \Delta u(t) \in f - N_K(u(t)) \quad \text{in } \Omega, \quad \text{for a.e. } t \in (0,T], \\ \dfrac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma_N, \\ u = 0 \ \text{ on } \Gamma_D, \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = v_0, \end{array}\right.$$

where Γ_N and Γ_D is a partition of $\partial\Omega$, Γ_D being of non zero measure in $\partial\Omega$. This models for instance the contact between an antiplane elastic structure with a rigid foundation or a stretched drum membrane under an obstacle condition.

We build now the approximation spaces thanks to finite element methods. Let $\mathscr{T}^h$ a regular triangular mesh of Ω (in the sense of Ciarlet [3], h being the diameter of the largest element) and W^h be the P_1+ finite element space

$$W^h = \{w^h \in \mathscr{C}^0(\Omega) : w^h = \sum_{a_i \in \mathscr{A}} w_i \varphi_i + \sum_{T \in \mathscr{T}^h} w_T \varphi_T\},$$

where $\mathscr{A}$ is the set of the vertices of the mesh which do not lie on Γ_D, φ_i, $i \in \mathscr{A}$ are the piecewise affine function satisfying

$$\varphi_i(a_j) = \delta_{ij},$$

i.e. the shape functions of a P_1 finite element method on $\mathscr{T}^h$. Each functions φ_T, $T \in \mathscr{T}^h$ is a cubic bubble function whose supports is T. Let H^h be the P_0 finite element space

$$H^h = \{v^h \in L^2(\Omega) : v^h = \sum_{T \in \mathscr{T}^h} v_T \mathbb{1}_T\},$$

and finally, let K^h be defined as

$$K^h = \{w^h \in W^h : w^h(a_i) \geq 0 \text{ for all } a_i \text{ vertex of } \mathscr{T}^h\}, \tag{14}$$

which means that the constraints are only prescribed at the vertices of the mesh.

Lemma 2 *This choice of W^h, H^h and K^h satisfies condition (7).*

Proof. The computation of F^h gives

$$\begin{aligned} F^h &= \{w^h \in W^h : \int_T w^h dx = 0 \ \ \forall T \in \mathscr{T}^h\} \\ &= \{w^h = \sum_{a_i \in \mathscr{A}} w_i \varphi_i + \sum_{T \in \mathscr{T}^h} w_T \varphi_T : w_T = -\int_T \sum_{a_i \in \mathscr{A}} w_i \varphi_i dx\}, \end{aligned}$$

while the functions g^i are defined by

$$g^i(w^h) = -w^h(a_i), \quad a_i \in \mathscr{A}.$$

Thus one has $g^i(w^h) = w_i$ and the surjectivity of G on F is obvious. $\square$

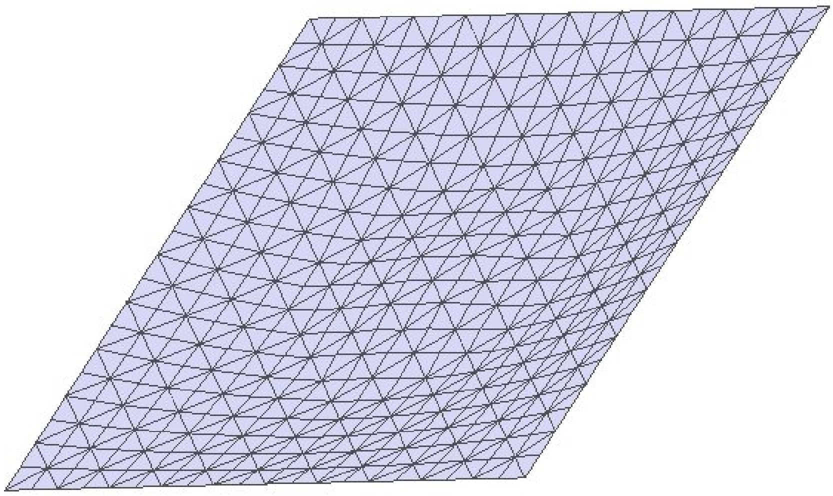

Figure 1: $h = 0.05$.

5 Numerical experiments

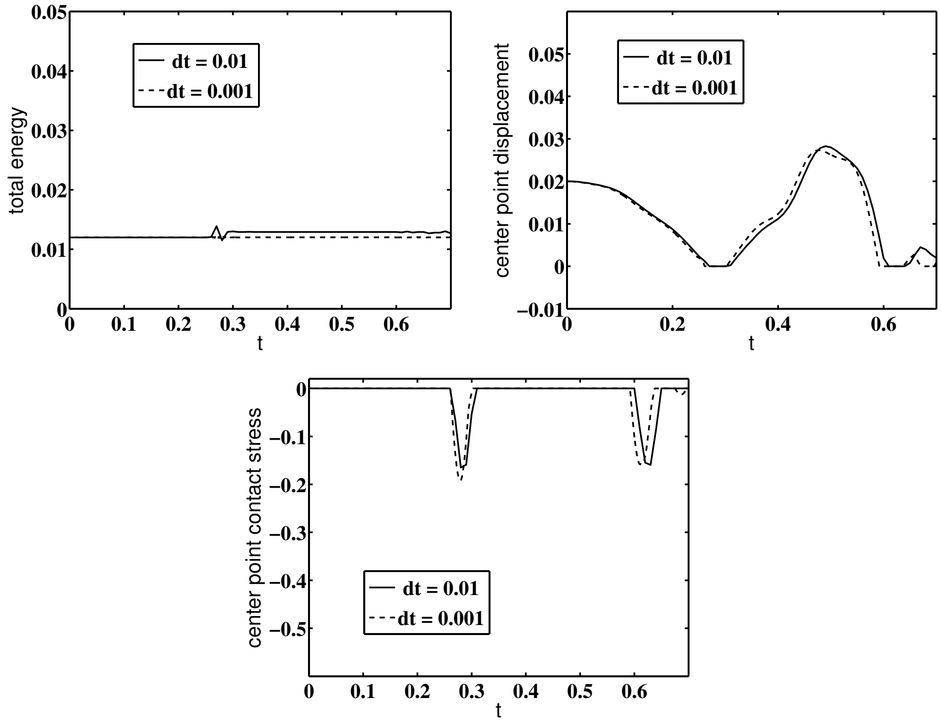

Figure 2: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_1+/P_0 *method, a midpoint scheme and with* $h = 0.1$.

We present now some numerical experiments done on the problem described in the previous section, with

$$\Omega = (0,1) \times (0,1), \quad \Gamma_D = \partial\Omega, \quad \Gamma_N = \emptyset, \quad f = -0.6.$$

The initial condition is

$$u(0,x) = 0.02, \quad \dot{u}(0,x) = 0, \quad x \in \Omega,$$

and we consider a non-homogeneous Dirichlet condition

$$u(t,x) = 0.02, \quad x \in \partial\Omega.$$

The structured mesh used can be viewed on Figure 1 where the solution is represented during the first impact on the obstacle. The numerical experiments are done using the finite element library Getfem++ [18]. The program is available on Getfem++ web site. All the numerical experiments use the same definition of K^h corresponding to (14).

The first numerical test is made with a midpoint scheme and the approximation presented in Section 4, that is a P_1+/P_0 method (P_1+ for u^h and P_0 for $\dot{u}^h$).

In good accordance with the theoretical results, the curves on Figure 2 show that the energy tends to be conserved when the time step decreases. Moreover, both the displacement and the contact stress taken at the point $(0.5, 0.5)$ are smooth and converge satisfactorily.

Conversely, the curves on Figures 3 and 4 obtained for a P_1/P_0 method and a P_1/P_1 method respectively are unstable since the energy is growing very fast after the first impact. The displacement and the contact stress are very oscillating and do not converge. Conversely, the instabilities are more important for the smallest time step. These two methods do not satisfy the condition (7) since $\dim(H^h) \geq \dim(W^h)$. Note that the P_1/P_1 method corresponds to a standard Galerkin approximation of the problem.

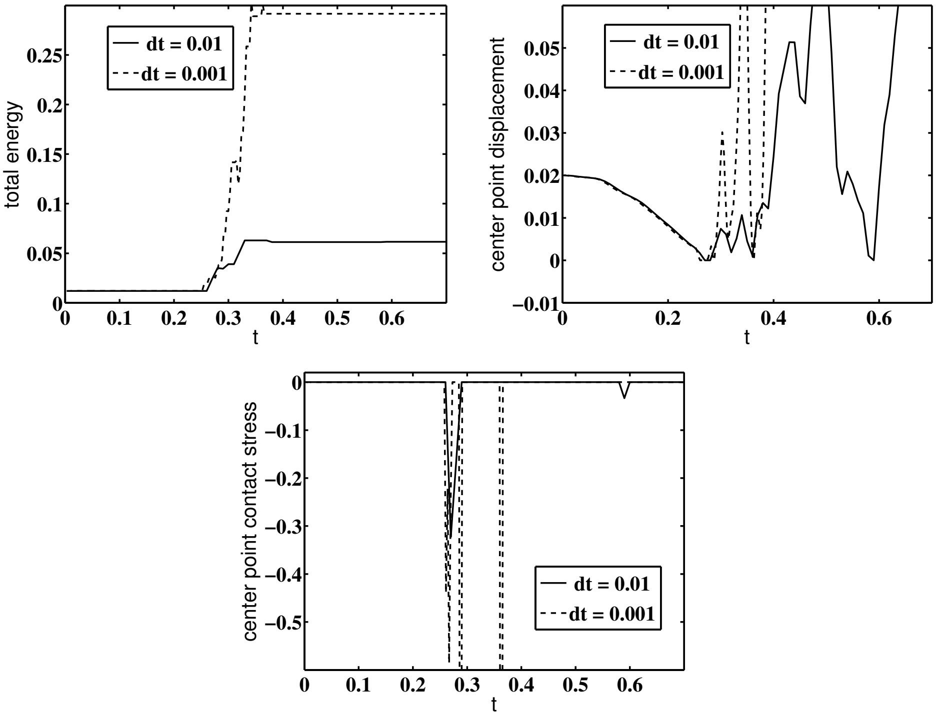

Figure 3: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_1/P_0 *method, a midpoint scheme and with* $h = 0.1$.

Even thought we do not have a proof that condition (7) is satisfied for P_1+/P_1 and P_2/P_1 methods (still with the same K^h), Figures 5 and 6 show that the midpoint scheme is stable and converging for these two methods. The fact that a stable method is obtained with the P_2/P_1 method tends to demonstrate that this strategy is not limited to order one approximation like the methods presented in [9, 4].

An interesting situation is also presented on Figures 7, 8 and 9 where a backward Euler scheme is used. This time integration scheme is unconditionally stable because it is possible to prove that the discrete energy decreases from an iteration to another (see [8] for instance). This is the case for any choice of W^h and H^h. Consequently, this method present some smooth results for the displacement and the contact stress. However, the energy decreases rapidly for large time steps. Figure 7 shows that for a well-posed method, the energy tends to be conserved for small time steps, but Figures 8 and 9 show that with an ill-posed method (such as classical discretizations) there is an energy loss at the impact which do not decay when the time step and the mesh parameter decrease. This means that with an ill-posed method, we do not approximate a physical solution of the problem whenever one expects energy conservation to be satisfied at the limit.

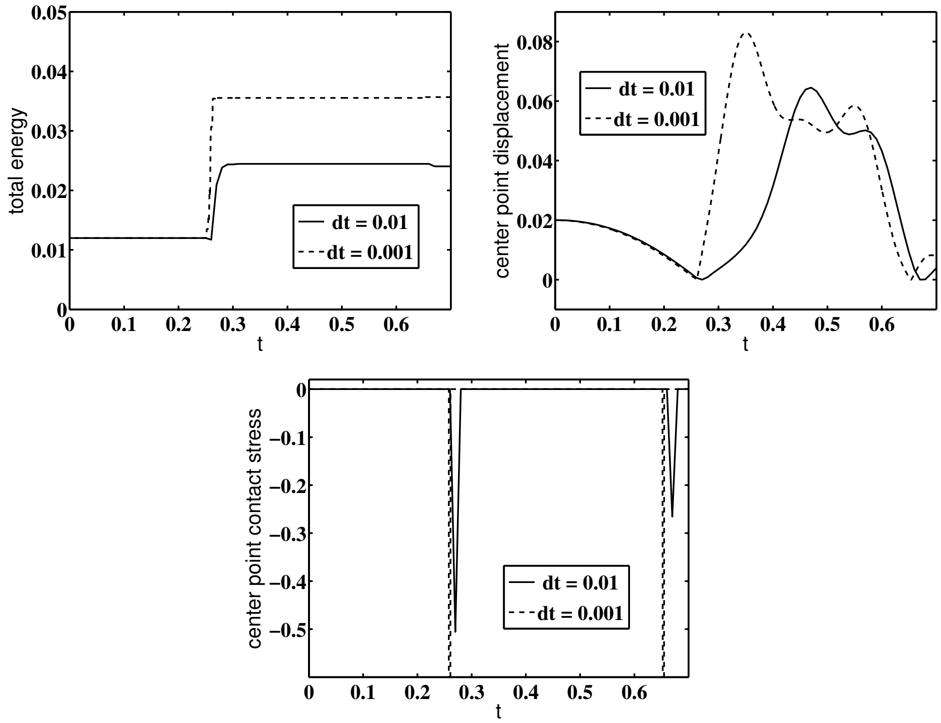

Figure 4: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_1/P_1 *method, a midpoint scheme and with* $h = 0.1$.

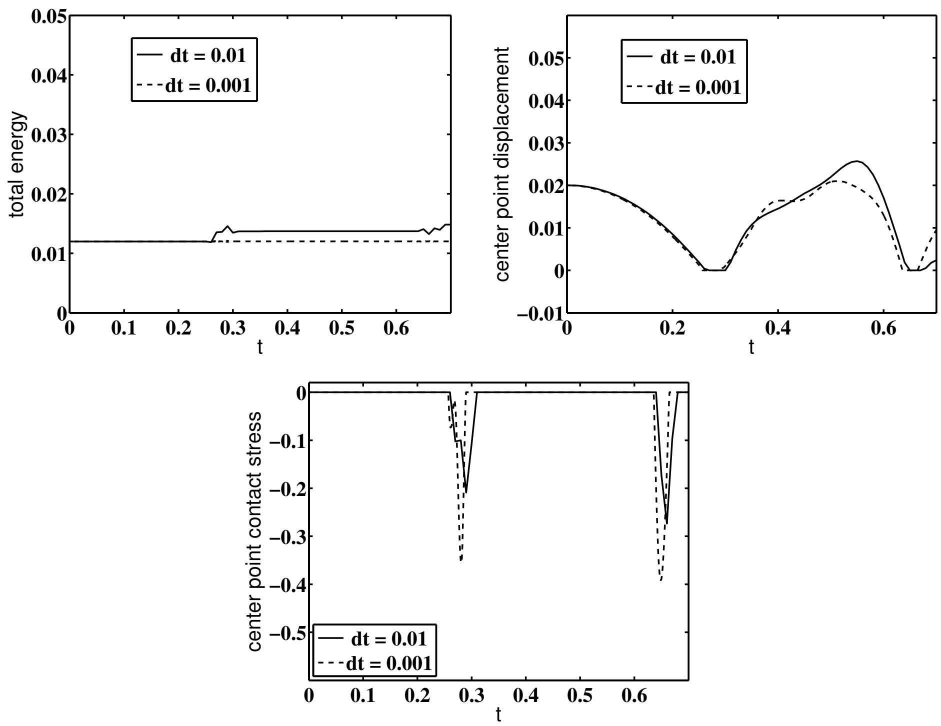

Figure 5: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_1+/P_1 *method, a midpoint scheme and with* $h = 0.1$.

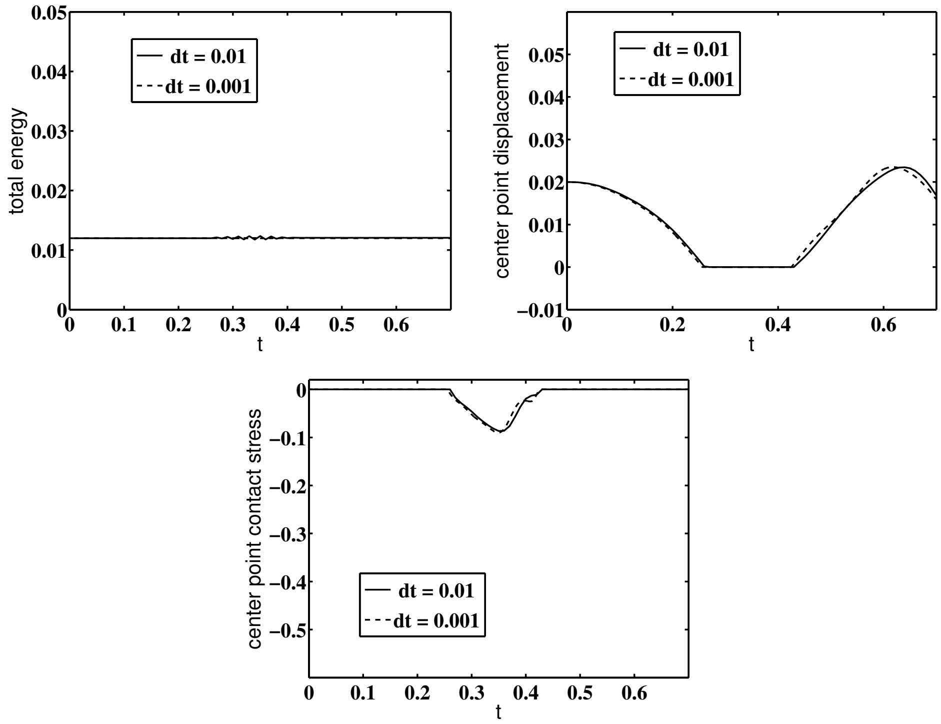

Figure 6: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_2/P_1 *method, a midpoint scheme and with* $h = 0.1$.

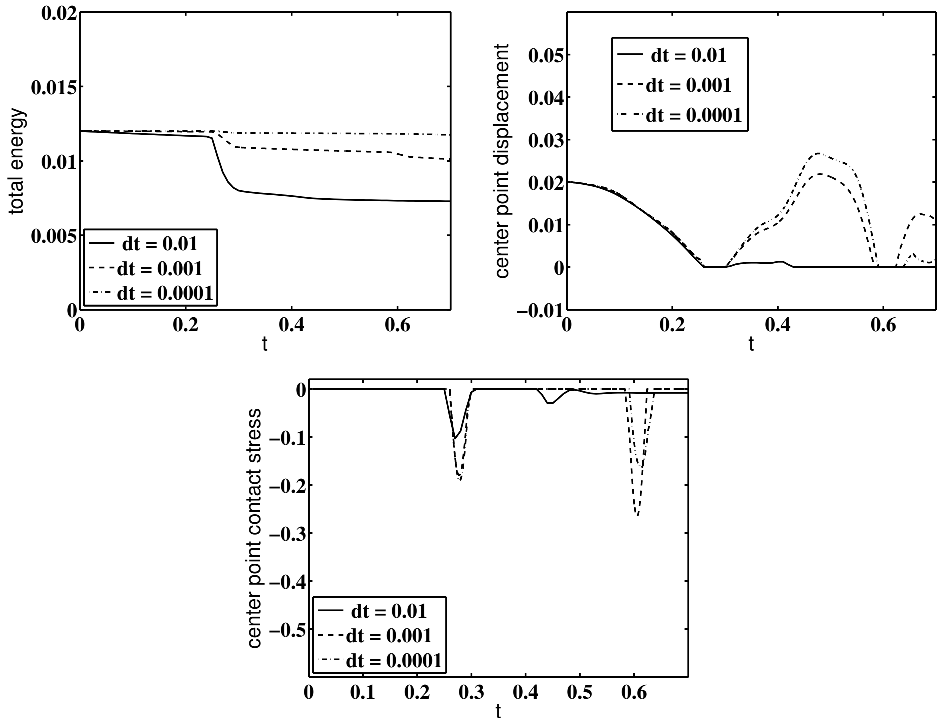

Figure 7: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* $P_1{+}/P_0$ *method, a backward Euler scheme and with* $h = 0.1$.

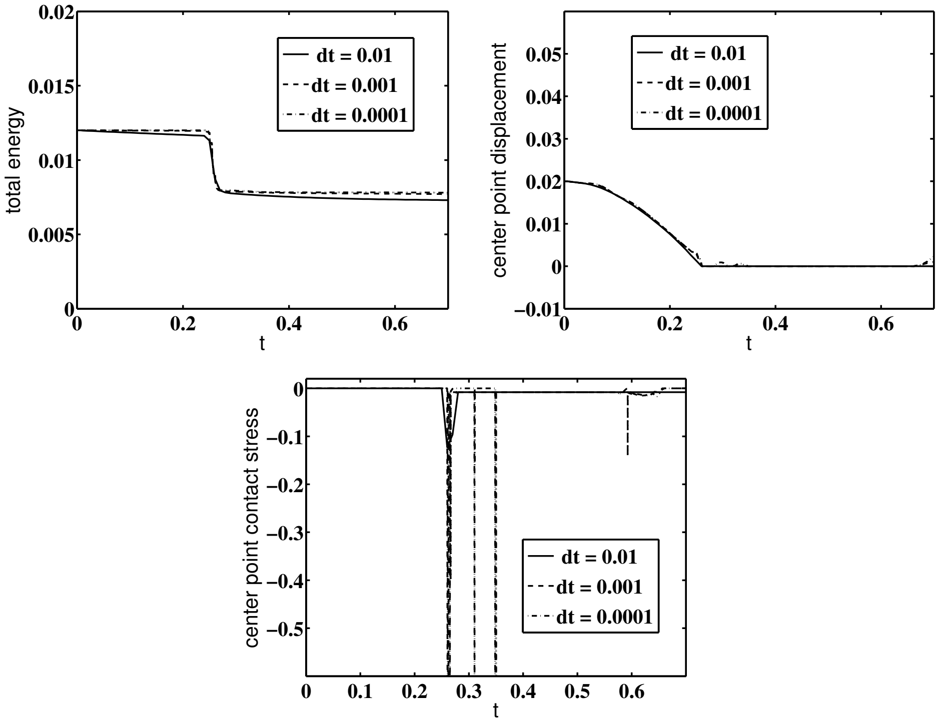

Figure 8: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_1/P_0 *method, a backward Euler scheme and with* $h = 0.1$.

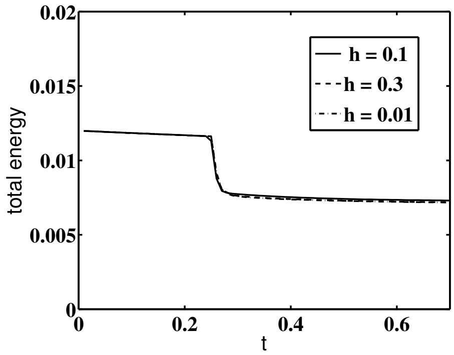

Figure 9: *Evolution of the energy, the displacement at the center point* $(0.5, 0.5)$ *and the contact stress at the center point for a* P_1/P_0 *method, a backward Euler scheme and with* $\Delta t = 0.001$.

6 Concluding remarks

The classical space semi-discretizations of second order hyperbolic problems with constraints leads to ill-posed problems. The great majority of time integration schemes are unstable when they are applied to such semi-discretizations. The proposed strategy allows to have well-posed semi-discretizations and ensure that the standard time integration schemes converge toward an energy conserving solution at least for a fixed space semi-discretization. Note that we did not discuss the overall stability of the full discretization, which is a perspective of this work.

The numerical experiments show that, even with stable time integration schemes, classical space semi-discretizations can lead to non physical solutions.

It should be possible to extend the analysis to more general set of constraints replacing the condition (7) by a condition on the tangent cone at each instant.

The advantages compared to the methods presented in [9, 4] is that arbitrary order methods can be obtained here and the strategy can be applied to thin structure.

Acknowledgments

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